Dynamical Systems

Exam 20 January 2025

The exam consists of 4 questions. You have 120 minutes to do the exam. You can achieve 50 points in total which includes a bonus of 5 points. The number of points per question are given in square brackets.

1. [**9 points in total**] Each of the following time-continuous systems depends on a parameter $a \in \mathbb{R}$.

   (a) For the following one-dimensional systems, sketch the bifurcation diagram including representative phase portraits and classify the bifurcations of equilibrium points.

      i. [**3 pts**] $x' = x\cos x + ax$,

      ii. [**3 pts**] $x' = x\sin x + ax$.

   (b) [**3 pts**] For the planar systems

$$r' = r - r^3,$$
$$\theta' = a + \sin\theta,$$

   where r and θ are polar coordinates, sketch representative phase portraits in the Cartesian coordinate plane and sketch the bifurcation diagram in a diagram θ versus a.

2. [**9 points**] Consider the planar systems

$$X' = \begin{pmatrix} 2a & b \\ -b & 0 \end{pmatrix} X$$

   with parameters $a, b \in \mathbb{R}$. Sketch the regions in the (a, b) plane where this system has different types of canonical forms. In each region give the canonical form and sketch the phase portrait of the system in canonical form.

3. [**15 points in total**] Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined as

$$f(x, y) = \frac{1}{2}y^2 - \frac{1}{3}x^3 + x$$

   for $(x, y) \in \mathbb{R}^2$.

   (a) Consider the gradient system $X' = F(X)$ with $F = -\nabla f$ and $X = (x, y) \in \mathbb{R}^2$.

      i. [**2 pts**] Show that the system has the equilibrium points $(x_-, y_-) = (-1, 0)$ and $(x_+, y_+) = (1, 0)$, and show from the linearization that (x_-, y_-) is asymptotically stable und that (x_+, y_+) is a saddle.

      ii. [**3 pts**] Sketch the phase portrait of the system including the stable und unstable curves of (x_+, y_+). (Hint: it can be helpful to also use the nullclines.)

– please turn over –

iii. **[3 pts]** Construct a strict Lyapunov function as $L = f + c$ for the equilibrium $(x_-, y_-) = (-1, 0)$ by choosing the real constant c and the domain of L in a suitable way to obtain the *full* basin attraction of $(x_-, y_-))$ from the Lyapunov Stability Theorem.

(b) Consider now the Hamiltonian system $X' = F(X)$ with $F = (f_y, -f_x)$ and $X = (x, y) \in \mathbb{R}^2$.

i. **[2 pts]** Show that the system has the equilbirium points $(x_-, y_-) = (-1, 0)$ and $(x_+, y_+) = (1, 0)$, and show from the linearization that (x_+, y_+) is a saddle.

ii. **[3 pts]** Sketch the phase portrait of the system including the stable und unstable curves of (x_+, y_+).

iii. **[2 pts]** Construct a Lyapunov function as $L = f + c$ for the equilibrium $(x_-, y_-) = (-1, 0)$ by choosing the real constant c and the domain of L in a suitable way to show that (x_-, y_-) is Lyapunov stable.

4. **[12 points in total]**

(a) **[9 pts]** Let $t : [0, 1] \to [0, 1]$ be defined as $t(x) = 3x \bmod 1$, $x \in [0, 1]$. Show that the discrete-time system $x_{n+1} = f(x_n)$ with $x_n \in [0, 1]$ for $n = 0, 1, 2, \ldots$, satisfies all three conditions of Devaney's definition of chaos.

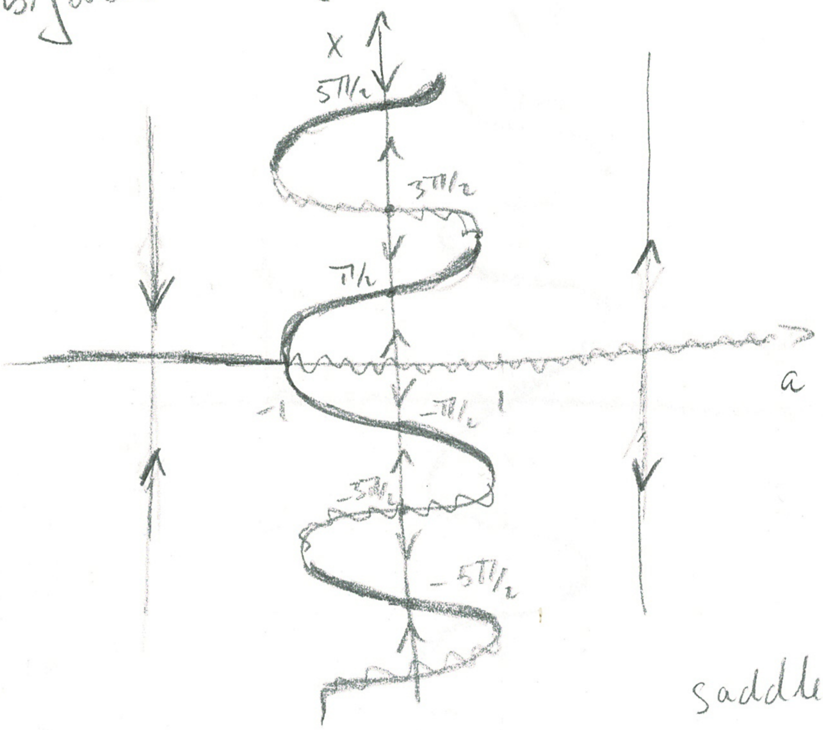
(b) **[3 pts]** Let I and J be compact intervals in $\mathbb{R}$. Show that if the discrete-time system $x_{n+1} = f(x_n)$ with $f : I \to I$ and $x_n \in I$ for $n = 0, 1, 2, \ldots$ has dense periodic points and is topologically conjugate to the discrete-time system $y_{n+1} = g(y_n)$ with $g : J \to J$ and $y_n \in J$ for $n = 0, 1, 2, \ldots$, then also the latter system has dense periodic points.

1. a) $x' = x\cos x + ax$

equilibria: $x\cos x + ax = 0$

$\Leftrightarrow x = 0 \cup a = -\cos x$

bifurcation diagram:



— : stable

~ : unstable

saddle node bifurcation

at $(a,x) = (1, (2n+1)\pi)$
$n \in \mathbb{Z}$

and $(a,x) = (-1, 2n\pi)$
$n \in \mathbb{Z}\setminus\{0\}$

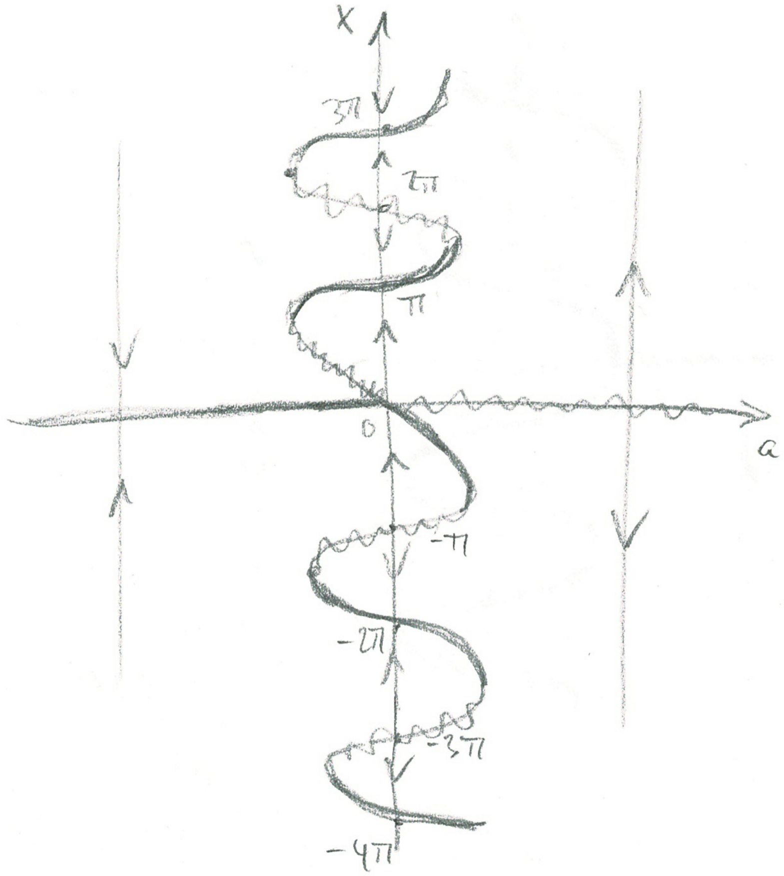
pitchfork at
$(a,x) = (-1,0)$

b) $x' = x \sin x + a x$

equilibria $x \sin x + a x = 0$

$\Leftrightarrow x = 0 \cup a = -\sin x$

bifurcation diagram:



saddle-node bifurcations at

$(a, x) = (1, (2n+1)\pi - \frac{\pi}{2})$

$n \in \mathbb{Z}$

$(a, x) = (-1, 2n\pi - \frac{\pi}{2})$

$n \in \mathbb{Z}$

transcritical at

$(a, x) = (0, 0)$

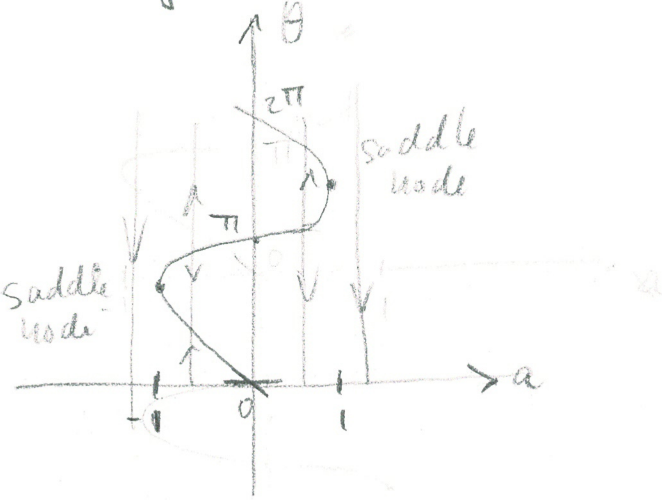
c) $r' = r - r^3$ $\qquad r \geq 0,\ \theta \in [0, 2\pi)$

$\theta' = a + \sin\theta$

equilibria: $r = 0$ or $r = 1$

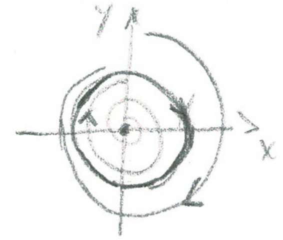
with $a + \sin\theta = 0$

bifurcation diagram:



phase portraits:

$a < -1$: $\theta' < 0$



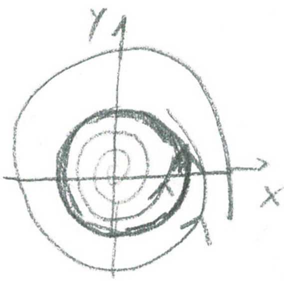
stable limit cycle

$r = 1$

spiral source at

$(x, y) = (0, 0)$

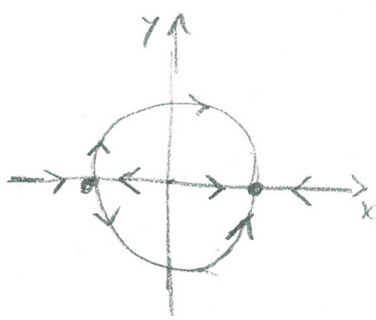
$a > 1$: $\theta' > 0$



same as above

$-1 < a < 1$

e.g. $a = 0$:



saddle

and

sink (node)

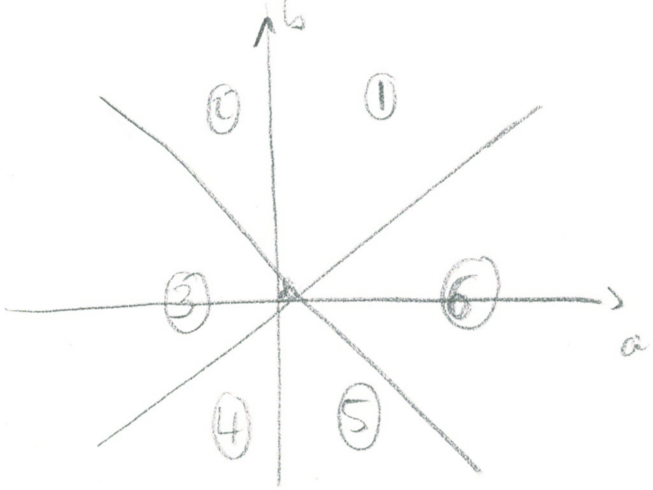
2. $X' = A\bar{X}$ with $A = \begin{pmatrix} 2a & b \\ -b & 0 \end{pmatrix}$

eigenvalues: $(2a-\lambda)(-\lambda) + b^2 = 0$

$\Leftrightarrow \lambda^2 - 2a\lambda = -b^2$

$\Leftrightarrow \lambda = a \pm \sqrt{a^2 - b^2}$

bifurcation diagram



① spiral source　② spiral sink

⑤ — " —　④ — " —

canonical form: $\begin{pmatrix} a & \sqrt{b^2-a^2} \\ -\sqrt{b^2-a^2} & a \end{pmatrix}$

③ real sink　④ real source

canonical form $\begin{pmatrix} a+\sqrt{a^2-b^2} & 0 \\ 0 & a-\sqrt{a^2-b^2} \end{pmatrix}$

3. $f(x,y) = \frac{y^2}{2} - \frac{1}{3}x^3 + x$

a) $$\begin{pmatrix} x' \\ y' \end{pmatrix} = - \begin{pmatrix} -x^2+1 \\ y \end{pmatrix} = \begin{pmatrix} x^2-1 \\ -y \end{pmatrix} =: F(x,y)$$

(i) equilibria: $y = 0 \wedge x^2 - 1 = 0$

$$\Leftrightarrow (x,y) = (\pm 1, 0) =: (x_\pm, y_\pm)$$

matrix associated with linearization:

$$DF = \begin{pmatrix} 2x & 0 \\ 0 & -1 \end{pmatrix}$$

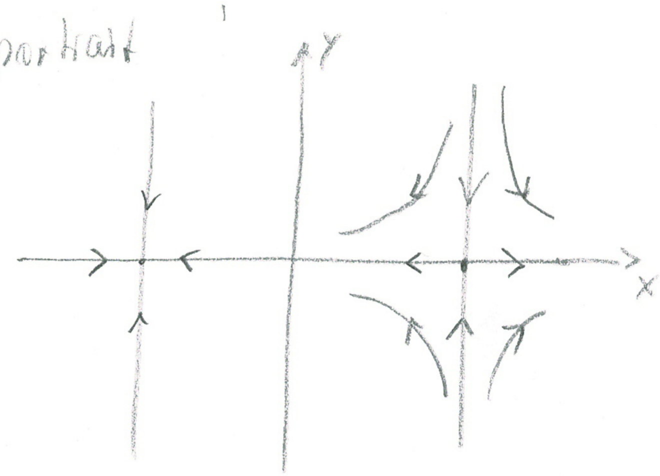
at (x_-, y_-): $DF = \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix}$ real sink (asymptotically stable)

at (x_+, y_+): $DF = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}$ saddle

(ii) x-nullclines: $x = \pm 1$

y nullclines: $y = 0$

phase portrait



$x = 1$: stable curve of saddle at (x_+, y_+)

$y = 0$, $x > x_-$: unstable curve of saddle at (x_+, y_+)

(iii) Set $c = -f(x_-, y_-) = -\left(\frac{1}{3} - 1\right) = \frac{2}{3}$

$\Rightarrow \quad L = f + c \geq 0$ on $x < x_+ = 1$

and $\dot{L} = -\nabla f \nabla f = -\|\nabla f\|^2 \leq 0$

$\|\nabla f(x,y)\| = 0 \Leftrightarrow \nabla f(x,y) = 0$

$\Leftrightarrow (x,y)$ equilibrium

Hence: $L = f + c$ with

$$L(x,y) = \frac{1}{2}y^2 - \frac{1}{3}x^3 + x + \frac{2}{3}$$

is strict Lyapunov function on $x < x_-$

$\Rightarrow \quad (x_-, y_-)$ asymptotically stable

and $x < x_-$ belongs to basin of attraction.

In fact this is the full basin of attraction as the boundary $x = x_+$ is invariant and does not belong to the basin of attraction ($x = x_+$ is the stable curve of the saddle (x_+, y_+))

b) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} f_y \\ -f_x \end{pmatrix} = \begin{pmatrix} y \\ x^2-1 \end{pmatrix} =: F(x,y)$

(i) equilibria: as in part (a)

$$(x,y) = (\pm 1, 0) =: (x_\pm, y_\pm)$$

matrix associated with linearization: $DF = \begin{pmatrix} 0 & 1 \\ 2x & 0 \end{pmatrix}$

at (x_-, y_-): $DF = \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix}$

eigenvalues $(-\lambda)^2 = -2$ center

$\Rightarrow \lambda_\pm = \pm i\sqrt{2}$

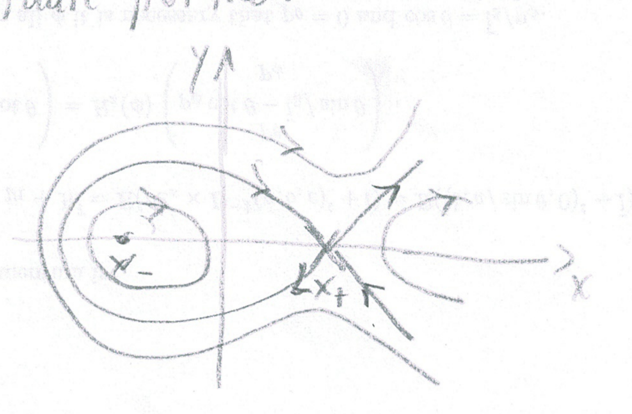
at (x_+, y_+): $DF = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$

eigenvalues: $(-\lambda)^2 = 2$

$\lambda_\pm = \pm\sqrt{2}$ saddle

(ii) solutions are contained in the level sets of f

hence phase portrait:



left branch of stable curve equals left branch of unstable curve

(iii) Like in part (a)

define L as

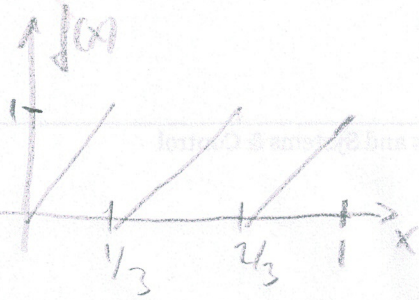
$$L = f + c \quad \text{with } c = \frac{2}{3}$$

$$\Rightarrow \dot{L} = f_x f_y - f_y f_x = 0$$

$\Rightarrow$ L Lyapunov function

$\Rightarrow$ (x_-, y_-) is Lyapunov stable

4. $f(x) = 3x \mod 1$, $x \in [0,1]$



Let $I^n_k = [(k-1)\frac{1}{3^n}, k\frac{1}{3^n}]$ $n = 1,2,3,\dots$ $k = 1,\dots,3^n$

then $f^{(n)}$ surjectively maps each I^n_k to $[0,1]$

note that each I^n_k has length $\frac{1}{3^n}$

(i) periodic points are dense

each I^n_k contains a periodic point of $f^{(n)}$

Let $x \in [0,1]$ and U be an open neighbourhood of x. To be shown: U contains a periodic point

As U is open there exists $\varepsilon > 0$ s.t. $(x-\varepsilon, x+\varepsilon) \subset U$. Choose n, k s.t. $I^n_k \subset (x-\varepsilon, x+\varepsilon) \subset U$. As I^n_k contains a periodic point, U contains a periodic point.

(ii) f is topol. transitive:

Let $U, V \subset [0,1]$ open.

To be shown: $\exists\, n \in \mathbb{Z}_{>0}$ s.t.

$f^{(n)}(U) \cap V \neq \emptyset$. Like in (ii)

$\exists\, n,k$ s.t. $I_k^n \subset U$. As $f(I_k^n) = [0,1]$

it follows that $f^{(n)}(I_k^n) \subset f^{(n)}(U) = [0,1]$

and hence $f^{(n)}(U) \cap V \neq \emptyset$

(iii) f has sensitive dependence on initial conditions: Choose β: $0 < \beta < \frac{1}{2}$ fixed.

Let $x \in [0,1]$ and U be open neighb. of x.

To be shown: $\exists\, y \in U$ and $n \in \mathbb{Z}_{>0}$ s.t.

$|f^{(n)}(x) - f^{(n)}(y)| > \beta$. As above $\exists\, n,k$

s.t. $I_k^n \subset U$. As $f^{(n)}(I_k^n) = [0,1]$

there exist $y \in I_k^n \subset U$ s.t.

$|f^{(n)}(y) - f^{(n)}(x)| > \beta$

b) Let $y_0 \in J$ and U be open neighb. of y_0. To be shown: $\exists y \in U$, with y being a periodic point.

Let $V = h^{-1}(U)$ where $h: I \to J$ is a homeomorphism with

$$h(f^{(n)}(x)) = g^{(n)}(h(x)) \text{ f.a. } x \in I$$

Such a function h exists by definition as f and g are top. conjugate.

Then V is open as h is continuous.

and $h^{-1}(y_0) =: x_0 \in V$. As f has dense periodic point $\exists x \in V$ periodic, i.e. $\exists n \in \mathbb{Z}_{>0}$ s.t. $f^{(n)}(x) = x$.

Set $y = h(x)$. $\Rightarrow$ $y \in U$ and

$$g^{(n)}(y) = g^{(n)}(h(x)) = h(f^{(n)}(x)) = h(x) = y$$

So y is periodic.

Here we used

$g^{(n)} \circ h = h \circ f^{(n)}$ which can be shown by induction.